DRAG OF A PLATE IN A UNIFORM FLOW OF A WEAK POLYMER SOLUTION

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The problem of the drag of a plate moving in a weak polymer solution is solved on the basis of a three-layer schematization of the velocity distribution in the turbulent boundary layer.

The presence of polymer macromolecules in a flow leads to a substantial reduction in turbulent friction [1]. The problem of the drag of a plate moving in a weak polymer solution is of both practical and theoretical interest, and a considerable amount of attention has been devoted to its solution [1]. The simplest reliable method of calculating the resistance characteristics of a plate is based on the solution of integral momenta equations. The effect of additives on turbulence is manifest in a change in the velocity profile in the boundary layer. In [2-5] use was made of a logarithmic distribution of dimensionless mean velocity

$$u^{+}=\frac{1}{\varkappa}\ln\eta+B, \tag{1}$$

in which the presence of additives led to an increase in the parameter B compared to its value $B \approx 5.5$ in a Newtonian fluid:

$$B=5.5+\alpha\ln\frac{v_*}{v_*^0}. \tag{2}$$

Information on different values of α and v_*^0 for different polymers is contained in [6-8].

In [2-5], Eqs. (1) and (2) were used to obtain approximate formulas for the local-drag coefficient of a plate c_f moving in a uniform polymer solution. In [9], a system of formulas more complicated than (2) was used to find parameter B and a numerical solution was obtained for a problem of plate drag.

An attempt was made in [2] to obtain an exact solution for integral momenta equations on the basis of (1) and (2), but an error was made during the transformations [1, 5] and the final formula for c_f lacks several necessary terms. There are two basic difficulties with the use of Eqs. (1) and (2) in drag calculations. It is known that polymer additives significantly increase the thickness of the buffer zone of the flow, which may spread over the entire thickness of the boundary layer, and the region of logarithmic velocity profile (1) may on the whole be absent. Moreover, Eq. (2) does not consider the existence of asymptotes of maximum drag reduction, which limit the amount of reduction in friction in polymer solutions (the value of B in (2) increases without limit with an increase in the parameter v_*/v_*^0).

In connection with the above, it is best to use the three-layer velocity profile schematization proposed in [10] in calculations of turbulent boundary layers in liquids with polymer additives. The flow region in the boundary layer is broken down into three zones – viscous sublayer, buffer zone, and external flow region:

$$u^{+}=\begin{cases}\eta, & \eta\leqslant\eta_0,\\ \dfrac{1}{\varkappa'}\ln\eta+B', & \eta_0<\eta\leqslant\eta_1,\\ \dfrac{1}{\varkappa}\ln\eta+B, & \eta_1<\eta\leqslant\eta_\delta,\end{cases} \tag{3}$$

where the parameters $\varkappa$, $\varkappa'$, B' and η_0 are constant ($\varkappa = 0.4$, $\varkappa' = 0.085$, $B' = -17.0$, $\eta_0 = 11.6$).

The work [11] presents results of numerous measurements of velocity in a turbulent flow of weak polymer solutions, making it possible to judge the advantages of scheme (3) compared to (1).

M. V. Lomonosov Moscow State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vo. 41, No. 5, pp. 819-826, November, 1981. Original article submitted January 9, 1980.

0022-0841/81/4105-1195$07.50

The following relation was used in [10] for the parameter η_1

$$\eta_1 = \begin{cases} \eta_0 \left(\dfrac{\eta_\delta}{\eta_\delta^0}\right)^{\Psi}, & \eta_\delta > \eta_\delta^0, \quad \eta_\delta = \dfrac{\delta v_*}{\nu}, \\ \eta_\delta, & \eta_\delta \leqslant \eta_0 \left(\dfrac{\eta_\delta}{\eta_\delta^0}\right)^{\Psi}, \quad \eta_\delta^0 = \dfrac{\delta v_*^0}{\nu}. \end{cases} \tag{4}$$

Equations (3) and (4) are in a certain sense a generalization of Eqs. (1) and (2): the values of drag calculated according to the three-layer scheme and scheme (1) and (2) coincide when $\eta_\delta \gg \eta_1$.

A solution is obtained below for a problem of the frictional resistance of a smooth plate in a flow of a weak polymer solution. Equations (3) and (4) are used to obtain the solution. It should be noted that [12] examined the drag of a plate on the basis of three-layer scheme (3). Here it was assumed that η_1 in (3) is a function of the concentration and molecular characteristics of the dissolved polymer but – in contrast to (4) – that it does not depend on the parameter v_*. The assumption employed in [12] regarding the parameter η_1 does not allow for consideration of the existence of a threshold value v_*^0 and an increase in the degree of drag reduction with an increase in the Reynolds number.

The integral momenta relation for a flat plate has the form [13]

$$\frac{d(\delta_2^+ v^+)}{dr} = \frac{1}{(v^+)^2}, \quad \delta_2 = \frac{1}{U^2}\int_0^\delta u(U-u)\,dy, \tag{5}$$

$$\delta_2^+ = \frac{\delta_2 v_*}{\nu}, \quad v^+ = u^+(\eta_\delta) = \frac{U}{v_*}, \quad r = \frac{Ux}{\nu}.$$

Polymer additives only slightly increase fluid viscosity at low concentrations and have almost no effect on the critical Reynolds number associated with the transition to turbulent flow. In connection with this, we will assume that there is a laminar boundary layer in the forward part of the plate and that the Blasius solution is valid for this layer. From the condition of continuity of the momentum thickness at the boundary of the regime transition, we obtain

$$0.664\sqrt{r_*} = \delta_2^+ v^+. \tag{6}$$

As in [2-5, 9], we will ignore the viscous sublayer in calculating the quantity δ_2^+ in the turbulent boundary layer.

Located beyond the section occupied by the laminar boundary layer are three regions of the turbulent boundary layer with different velocity distributions.

In the region satisfying the inequality $\eta_\delta \leq \eta_0(\eta_\delta/\eta_\delta^0)^{\Psi}$, the effect of the polymer additives will spread over the entire thickness of the boundary layer. The velocity profile in this region is approximated by a logarithmic distribution with the parameters $\varkappa'$ and B'. The following formula holds for δ_2^+

$$\delta_2^+ = \frac{1}{(v^+)^2}\int_0^{\eta_\delta}\left(\frac{1}{\varkappa'}\ln\eta + B'\right)\left(v^+ - \frac{1}{\varkappa'}\ln\eta - B'\right)d\eta = \frac{\eta_\delta}{\varkappa' v^+}\left(1 - \frac{2}{\varkappa' v^+}\right). \tag{7}$$

At $\eta_\delta > \eta_\delta^0$, the effect of the polymer additives will be manifest in an increase in the thickness of the buffer region of the flow. The velocity profile in this case consists of two sections of logarithmic distributions:

$$(u^+ = (\varkappa')^{-1}\ln\eta + B', \ \eta \leqslant \eta_1; \ u^+ = \varkappa^{-1}\ln\eta + B, \ \eta_1 < \eta \leqslant \eta_\delta).$$

The momentum thickness is expressed in the form

$$\delta_2^+ = \frac{\eta_\delta}{\varkappa v^+}\left(1 - \frac{2}{\varkappa v^+}\right) - \frac{\eta_1}{(v^+)^2}\left(\frac{1}{\varkappa'} - \frac{1}{\varkappa}\right)\left(v^+ - 2u_1^+ + \frac{2}{\varkappa'} + \frac{2}{\varkappa}\right),$$
$$u_1^+ = \frac{1}{\varkappa'}\ln\eta_1 + B'. \tag{8}$$

At $\eta_\delta \leq \eta_\delta^0$, the polymer additives have no effect on the turbulent velocity profile, and the dimensionless momentum thickness is calculated from the formula

$$\delta_2^+ = \frac{\eta_\delta}{\varkappa v^+}\left(1 - \frac{2}{\varkappa v^+}\right). \tag{9}$$

The dimensionless thickness of the boundary layer η_δ can be determined by means of the "friction law" which follows from (3) at $\eta = \eta_\delta$:

$$v^+ = \begin{cases} \frac{1}{\varkappa'} \ln \eta_\delta + B', & \eta_\delta \leqslant \eta_0 (\eta_\delta/\eta_\delta^0)^{\Psi}, \\ \frac{1}{\varkappa} \ln \eta_\delta + B, & \eta_\delta > \eta_\delta^0, \\ \frac{1}{\varkappa} \ln \eta_\delta + B_0, & \eta_\delta \leqslant \eta_\delta^0. \end{cases} \tag{10}$$

The following relation holds for parameter B in (10)

$$B = B' + \left(\frac{1}{\varkappa'} - \frac{1}{\varkappa}\right) \ln \eta_1 = B_0 + \left(\frac{1}{\varkappa'} - \frac{1}{\varkappa}\right) \Psi \ln \frac{v_*}{v_*^0}, \tag{11}$$

$$B_0 = B' + \left(\frac{1}{\varkappa'} - \frac{1}{\varkappa}\right) \ln \eta_0.$$

Comparing (2) and (11), we find the relation between α and Ψ at which the values of parameters B in Eqs. (1) and (3) coincide:

$$\alpha = \left(\frac{1}{\varkappa'} - \frac{1}{\varkappa}\right) \Psi.$$

It must be noted that, depending on the relation between the parameters r, $v_0^+ = U/v_*^0$, and α, the boundary-layer regions in which Eq. (7) or Eqs. (7) and (8) are valid for determining δ_2^+ may be absent.

Substituting the quantity η_δ from (10) into (7)-(9) and the latter into (5), we obtain an analytical solution of the problem of the frictional resistance of a smooth plate moving in a uniform polymer solution.

At $\eta_\delta \le \eta_0(\eta_\delta/\eta_\delta^0)^{\Psi}$, the quantity v^+ is related to the parameter r by the formula

$$r = \frac{1}{(\varkappa')^3} \exp\left[\varkappa' (v^+ - B')\right]\left[(\varkappa' v^+)^2 - 4\varkappa' v^+ - 6\right] + S_1. \tag{12}$$

At $\eta_\delta > \eta_\delta^0$, the following relation is valid

$$r = A_1 + A_2 + S_2, \tag{13}$$

$$A_1 = \left(\frac{v_*^0}{U}\right)^{\alpha\varkappa} \exp(-\varkappa B_0)\left[C_2 + \frac{\alpha\varkappa - 2}{\varkappa} C_1 - 2\,\frac{\alpha\varkappa - 1}{\varkappa^2} C_0\right],$$

$$A_2 = \begin{cases} \eta_1 v^+ \left(\frac{1}{\varkappa'} - \frac{1}{\varkappa}\right)\left[\frac{\Psi}{2-\Psi} - 2\,\frac{\Psi+1}{1-\Psi}\left(u_1^+ - \frac{1}{\varkappa}\right) + \right. \\ \qquad \left. + \frac{2 - 4\Psi - 2\Psi^2}{(1-\Psi^2)\varkappa'}\right], \quad \Psi \neq 1, \ \Psi \neq 2, \\ \eta_1 v^+ \ln v^+ \left(\frac{1}{\varkappa'} - \frac{1}{\varkappa}\right)\left[\frac{4}{\varkappa} + \frac{2}{\varkappa'} - 4u_1^+ + \right. \\ \qquad \left. + \frac{v^+}{\ln v^+} - \frac{\ln v^+}{2\varkappa'}\right], \quad \Psi = 1, \\ \eta_1 v^+ \left(\frac{1}{\varkappa'} - \frac{1}{\varkappa}\right)\left[2v^+ \ln v^+ + 6\left(u_1^+ - \frac{1}{\varkappa}\right) - \frac{7}{\varkappa'}\right], \\ \qquad \Psi = 2, \end{cases}$$

$$C_k = (v^+)^{\alpha\varkappa+k+1} \sum_{n=0}^{\infty} \frac{(\varkappa v^+)^n}{n!\,(\alpha\varkappa + h + k + 1)} \quad (k = 0,\ 1,\ 2).$$

For integral values $\alpha\varkappa$ = m, we will have

$$C_k = \exp(\varkappa v^+)\left[\frac{(v^+)^m}{\varkappa} + (-1)^{m+k}\frac{(m+k)!}{\varkappa^{m+k+1}} + \sum_{n=1}^{m+k+1}\frac{(m+k)!(v^+)^{m+k-n}}{(m+k-n)!\varkappa^{n-1}}\right].$$

At $\eta_\delta \le \eta_\delta^0$, we obtain a relation connecting r with v^+ which is valid for the turbulent boundary layer of a Newtonian fluid:

$$r = \frac{1}{\varkappa^3}\exp[\varkappa(v^+ - B_0)][(\varkappa v^+)^2 - 4\varkappa v^+ + 6] + S_3. \tag{14}$$

The above formulas also allow us to solve the problem of the development of a turbulent boundary layer on a plate on the basis of scheme (1) and (2). In this case, we should set $\eta_1 = 0$ in Eqs. (8). The relation between the parameters r and v^+ in this case will have the form

$$r = A_1 + S_4. \tag{15}$$

The parameters S_i(i = 1, 2, 3, 4) in (12)-(16) are constants of integration. They are found from the condition of continuity of the momentum thickness at the boundaries of the regions in which different formulas are used for the mean-velocity profiles.

Leaving the terms containing the higher power of v^+ in the formula for A_1 (see (13)) and using (15), we obtain the approximate relation

$$r = \frac{1}{\varkappa}\left(\frac{v_*^0}{U}\right)^{\alpha\varkappa}(v^+)^{\alpha\varkappa+2}\exp[\varkappa(v^+ - B_0)] + S_4. \tag{16}$$

Equation (16) was obtained in [2] with $S_4 = 0$ (i.e., without allowance for the region of the laminar boundary layer).

Figure 1 shows results of calculation of the local-drag coefficient of the plate $c_f = 2/(v^+)^2$. The polymer additives reduce turbulent drag on the forward part of the plate at $v_* > v_*^0$. The greater the value of the parameter α, the more substantial the effect of the additives in reducing turbulent drag. Each value of U/v_*^0 corresponds to a certain value of r_0 at which the additives cease to have an effect on turbulence. The coordinate r_0 can be determined by substituting the value of $v^+ = v_0^+$ into (14).

Figure 2 shows results of calculation of the total-drag coefficient C_F of the plate in relation to the Reynolds number $R = UL/\nu$ (L is the length of the plate):

$$C_F = \frac{1}{L}\int_0^L c_f dx = \frac{2}{L}\int_0^L \frac{d\delta_2}{dx}dx = \frac{2\delta_2(L)}{L}.$$

The drag coefficient begins to decrease when the condition $v_*(L) > v_*^0$ is satisfied, with the size of the reduction increasing with an increase in velocity. At a sufficiently high velocity, the drag curve joins up with the maximum-drag-reduction asymptote.

With the flow of a weak polymer solution about a smooth plate, the boundary layer is smaller compared to the case of motion of the plate in pure solvent. Figure 3 shows the effect of parameter α on the thickness of the boundary layer.

Table 1 shows the results of calculations obtained according to scheme (3) and (4), scheme (1) and (2), approximate formula (16) at $S_4 = 0$, and the method of conversion of the flow characteristics of a liquid with polymer additives based on the concept of an "effective" Reynolds number [2]. We used the values $\alpha = 15$ and $R/v_0^+ = 2.5\cdot 10^4$ in the calculations. It is apparent from the data in Table 1 that, at high Reynolds numbers, calculations based on Eqs. (1) and (2) yield lower values of plate drag coefficient than are obtained with calculations based on scheme (3) and (4). The accuracy of the approximate formulas for calculating C_F decreases considerably with an increase in the Reynolds number.

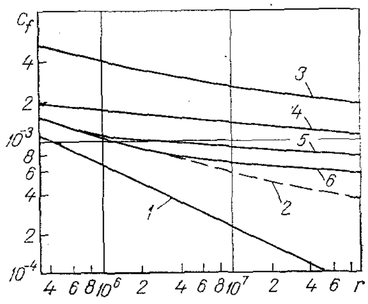

Fig. 1

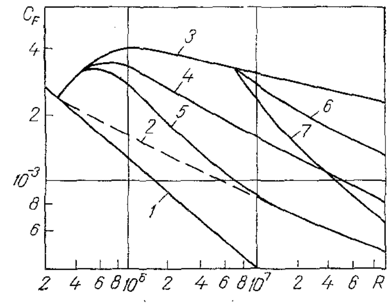

Fig. 2

Fig. 1. Dimensionless local-drag coefficient of plate at $v_0^+ = 400$, $r_* = 3 \cdot 10^5$; 1) drag coefficient with laminar regime of flow about the plate ($c_f = 0.664/\sqrt{r}$); 2) asymptote of maximum drag reduction obtained on the basis of the logarithmic velocity profile ($\varkappa' = 0.085$, $B' = -17.0$); 3) $\alpha = 0$; 4) 5; 5) 10; 6) 15.

Fig. 2. Total-drag coefficient of plate: 1) case of laminar flow about plate ($C_F = 1.328/\sqrt{R}$); 2) asymptote of maximum drag reduction; 3) $\alpha = 0$; 4) $\alpha = 5$, $R/v_0^+ = 2.5 \cdot 10^4$; 5) 15 and $2.5 \cdot 10^4$; 6) 5 and $2.5 \cdot 10^5$; 7) 15 and $2.5 \cdot 10^5$.

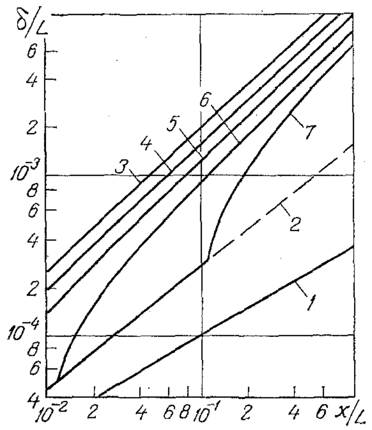

Fig. 3. Effect of polymer additives on the thickness of the turbulent boundary layer at $v_0^+ = 10^4$ and $R = 5 \cdot 10^7$: 1) dimensionless thickness of laminar boundary layer, $\delta/L = 5.0x/(L/\sqrt{r})$; 2) δ/L in the case of maximum effect of polymer additives on turbulence; 3) $\alpha = 0$; 4) 2.5; 5) 5; 6) 10; 7) 15.

TABLE 1. Data on Calculation of Total-Drag Coefficient of Plate at $\alpha = 15$ and $R/v_0^+ = 2.5 \cdot 10^4$

Reynolds number R	Drag coefficient of plate $C_F \cdot 10^3$			
	scheme (3) and (4)	scheme (1) and (2)	Eq. (16)	Eq. (22) in [2]
10^6	2,6476	2,6789	2,2222	2,9006
$3,16 \cdot 10^6$	1,4289	1,4235	1,1130	1,2925
10^7	0,8776	0,7730	0,6286	0,5760
$3,16 \cdot 10^7$	0,6036	0,4397	0,4164	0,2567

Thus, the results obtained show that, to improve the accuracy of calculations of turbulent drag in the flow of liquids with polymer additives, it is necessary to allow for the increase in the thickness of the buffer region of the boundary layer and the existence of the maximum-drag reduction asymptote.

NOTATION

u, longitudinal component of velocity; τ_W, friction stress on the wall; v_*, dynamic velocity; y, distance from wall; $\varkappa$, $\varkappa'$, B, B_0, B', parameters of the logarithmic velocity profile; ν, kinematic viscosity of the solution; ρ, density; $u^+ = u/v_*$, dimensionless longitudinal component of velocity; $\eta = yv_*/\nu$, dimensionless distance from wall; α, Ψ, parameters characterizing the effect of the additives on turbulence; v_*^0, threshold value of dynamic velocity; η_0, dimensionless thickness of viscous sublayer; η_1, boundary of buffer zone; δ, thickness of boundary layer; $\eta_\delta = \delta v_*/\nu$, dimensionless thickness of boundary layer; $\eta_\delta^0 = \delta v_*^0/\nu$; δ_2, momentum thickness; $\delta_2^+ = \delta_2 v_*/\nu$; $v^+ = u^+(\eta_\delta) = U/v_*$; $u_1^+ = u^+(\eta_1)$; $v_0^+ = U/v_*^0$; U, velocity of liquid at external boundary of boundary layer; x, longitudinal coordinate; $r = Ux/\nu$, local Reynolds number; r_*, critical value of Reynolds number associated with transition from laminar to turbulent flow regime; S_1, S_2, S_3, S_4, constants of integration; c_f, local-drag coefficient; C_F, total-drag coefficient; L, length of plate; $R = UL/\nu$, Reynolds number obtained in accordance with plate length.

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